Homological Algebra Seminar Week 8

Haotian Lyu after the talk of
Ballard William Roberts and Blum Milo Nicolas Jacques

3.4 Applications of the Universal Coefficient Theorem

Example 3.17. Singular Homology of a Topological Space X

Let $S(X) \in Ch(\mathcal{A}b)$ be the singular complex of X, then each $S_n(X)$ is a free abelian group. For $M \in \mathcal{A}b$, define homology of X with coefficients in M:

$$H_\bullet(X; M) := H_\bullet(S(X) \otimes M)$$

By the Universal Coefficient Theorem for Homology, for each n, we have the following:

$$H_n(X; m) \cong H_n(X; \mathbb{Z}) \otimes M \oplus Tor_1^{\mathbb{Z}}(H_{n-1}(X; \mathbb{Z}), M).$$

Theorem 3.18 (Künneth formula for complexes)**.** *Let R be a PID, $P_\bullet \in Ch(mod_R)$, then each P_n is free. Let $Q_\bullet \in Ch({}_R mod)$. Then for each n there exists a split short exact sequence:*

$$0 \to \bigoplus_{p+q=n} H_p(P) \otimes_R H_q(Q) \to H_n(P \otimes_R Q) \to \bigoplus_{p+q=n-1} Tor_1^R(H_p(P), H_q(Q)) \to 0.$$

Example 3.19. Let X, Y be topological spaces. Consider $X \times Y$. **Eilenberg Zilber Theorem** gives the following isomorphism:

$$H_\bullet(S(X \times Y)) \cong H_\bullet(S(X) \otimes S(Y)).$$

Then by 3.18, we have

$$H_n(X \times Y; R) \cong \left[\bigoplus_i (H_i(X; R) \otimes_R H_{n-i}(Y; R))\right] \oplus \left[\bigoplus_i Tor_1^R(H_i(X; R), H_{n-i-1}(Y; R))\right]$$

When R is a field, Tor groups vanish and we obtain the following isomorphism:

$$H_n(X \times Y; R) \cong \bigoplus_i (H_i(X; R) \otimes_R H_{n-i}(Y; R)).$$

4 Spectral Sequences

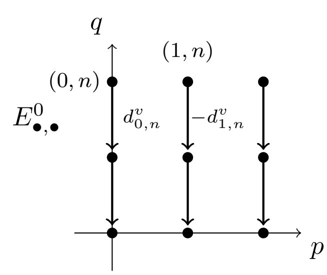
Our goal is to compute homology groups in a systematic way. Generally, a spectral sequence is a sequence of “pages”, and each “page” has a grid of objects that approximate homology. Move from “page n” to “page $n+1$” by calculating homology.

4.1 Introduction

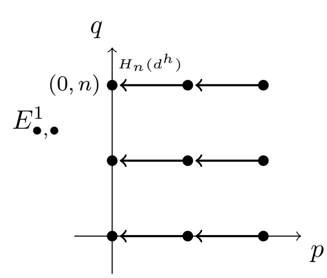
Begin with the following example.

Example 4.1. Let $E_{\bullet,\bullet}$ be a 1^{st} quadrant double complex. To compute the homology of $Tot_\bullet(E)$, use the following spectral sequence.

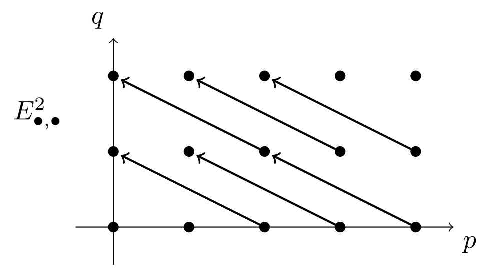
Define “**page 0**” as follows: $E^0_{p,q} := E_{p,q}$, $d^0_{p,q} := (-1)^p d^v_{p,q}$.



Each column is a chain complex, thus we can compute vertical homology at each point.
Then define “**page 1**” to be $E^1_{p,q} := H_q(E^0_{p,\bullet})$. For each p, $d^h_{p,\bullet} : E^0_{p,\bullet} \to E^0_{p-1,\bullet}$ is a morphism of chain complexes, then the morphisms in “page 1” are naturally defined to be $d^1_{p,q} := H_q(d^h_{p,\bullet}) : H_q(E^0_{p,\bullet}) \to H_q(E^0_{p-1,\bullet})$.



Each row is a chain complex. At each point, compute horizontal homology and define "**page 2**" as follows: $E^2_{p,q} := H_p(E^1_{\bullet,q})$.



Define morphism $d^2_{\bullet,\bullet}$ as the first exercise in the sheet 8 or the exercise 5.1.2 in the textbook, so that each line of slope $-\frac{1}{2}$ is chain complex. Repeat this process and we can define "**page n**".
To see why we are doing this, and how this spectral sequence is an approximation of homology of $Tot(E_{\bullet,\bullet})$, we will see another example.

Example 4.2. Suppose the only non-zero columns of $E_{\bullet,\bullet}$ are $A_\bullet = E_{0,\bullet}$ and $B_\bullet = E_{1,\bullet}$. Then the above spectral sequence computes $H_\bullet(Tot(E))$ up to extension in the sense that ∀n there is a short exact sequence

$$0 \to E^2_{0,2} \to H_n(T) \to E^2_{1,n-1} \to 0.$$

Proof. First, calculate $E^2_{0,n}$ and $E^2_{1,n-1}$:

$$E^2_{0,n} = H_0(E^1_{\bullet,n}) = coker(d^1_{1,n})$$

$$E^2_{1,n-1} = H_1(E^1_{\bullet,n-1}) \cong ker(d^1_{1,n-1}).$$

For each n, there is a short exact sequence:

$$0 \to \underbrace{E_{0,n}}_{A_n} \xrightarrow{i_n} \underbrace{E_{0,n} \oplus E_{1,n-1}}_{Tot(E)_n} \xrightarrow{\pi_n} \underbrace{E_{1,n-1}}_{B[-1]_n} \to 0$$

and thus the short exact sequence of chain complexes:

$$0 \to A \xrightarrow{i} Tot(E) \xrightarrow{\pi} B[-1] \to 0,$$

yielding the following long exact sequence:

$$\cdots \to \underbrace{H_{n+1}(B[-1])}_{H_n(B)} \xrightarrow{\partial_n} H_n(A) \xrightarrow{\widetilde{i_n}} H_n(Tot(E)) \xrightarrow{\widetilde{\pi_n}} \underbrace{H_n(B[-1])}_{H_{n-1}(B)} \to \dots$$

Thus there exists an exact short sequence for each n:

$$0 \to coker(\partial_n) \xrightarrow{\widetilde{i_n}} H_n(Tot(E)) \xrightarrow{\widetilde{\pi_n}} \underbrace{im(\widetilde{\pi_n})}_{ker(\partial_{n-1})} \to 0$$

Now we claim $\partial_n = H_n(d^h_{1,\bullet})$. In fact, if $[b] \in H_n(B)$, then $\widetilde{\pi}_{n+1}(0,[b]) = [b]$, and $d^{Tot}_{n+1}(0,b) = (d^h_{1,n}(b), d^v_{1,n}(b)) = (d^h_{1,n}, 0)$. Since $i_n(d^h_{1,n}) = (d^h_{1,n}, 0)$, therefore $\partial_n([b]) = [d^h_{1,n}] = H_n(d^h_{1,\bullet})([b])$. Hence the claim holds and we finish the proof. ∎

4.2 Homology Spectral Sequences and Cohomology Spectral Sequences

Definition 4.3. A homology spectral sequence (starting with E^a) in an abelian category $\mathcal{A}$ consists of the following data:

1. For each $r \geq a$, a family $\{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$ of objects in $\mathcal{A}$;
2. For each $r \geq a$, a family $\{\ d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\ \}$ of morphisms in $\mathcal{A}$, such that $d^r d^r = 0$, i.e. each line of slope $-\frac{r-1}{r}$ is a chain complex;
3. For each $r \geq a$, $\forall p,q \in \mathbb{Z}$, we have $E^{r+1}_{p,q} \cong ker(d^r_{p,q})/im(d^r_{p+r.q-r+1})$.

The **total degree** of $E^r_{p,q}$ is $p+q$. Each $d^n_{p,q}$ decreases the total degree by 1.

Definition 4.4. Let E, E' be spectral sequences over $\mathcal{A}$, a morphism $f : E \to E'$ is a family of morphisms in $\mathcal{A}$: $f^r_{p,q} : E^r_{p.q} \to E'^r_{p,q}$ where r is suitably large such that $d^r f^r = f^r d^r$ and $f^{r+1}_{p,q}$ is induced by $f^r_{p,q}$ on homology.

Remark 4.5. There is a category of homology spectral sequences.

Dually we can define **cohomology spectral sequence**.

Lemma 4.6 (Mapping lemma)**.** *Let $f : E \to E'$ be a morphism between two spectral sequences such that for some fixed r and each pair p,q, $f^r_{p,q}$ is an isomorphism. Then for each $s \geq r$, $f^s_{p,q}$ is an isomorphism.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
B^r_{p,q} \arrow[r] \arrow[d, "f^r_{p,q}"'] & Z^r_{p,q} \arrow[r] \arrow[d, "f^r_{p,q}"'] & E^{r+1}_{p,q} \arrow[r] \arrow[d, "f^{r+1}_{p,q}"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
B'^r_{p,q} \arrow[r] & Z'^r_{p,q} \arrow[r] & E'^{r+1}_{p,q} \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

By five lemma, $f^{r+1}_{p,q}$ is an isomorphism. By induction, for each $s \geq r$, $f^s_{p,q}$ is an isomorphism. ☐

Definition 4.7. A homology spectral sequence E is bounded if for each n, there are finitely many non zero terms of total degree n in $E^a_{\bullet,\bullet}$.

Lemma 4.8. *If E is bounded, for each p and q, there exists r_0 such that $E^r_{p,q} \cong E^{r_0}_{p,q}$ for every $r \geq r_0$.*

Proof. First notice that if $E^a_{p,q} = 0$ for some p,q, then by definition and induction we have for each $r \geq a$, $E^r_{p,q} = 0$.
For each fixed p,q, choose r_0 large enough such that $p+r_0$ is sufficiently large and $p-r_0$ is sufficiently small, such that for every $r \geq r_0$, $E^a_{p+r,q-r+1} = E^a_{p-r,q+r-1} = 0$, thus for each $r \geq r_0$, $E^r_{p+r,q-r+1} = E^r_{p-r,q+r-1} = 0$.
For each $r \geq r_0$ consider the chain complex

$$\cdots \to E^r_{p+r,q-r+1} \to E^r_{p,q} \to E^r_{p-r,q+r-1} \to \cdots$$

we have $E^{r+1}_{p,q} \cong E^r_{p,q}$. Thus $E^r_{p,q} \cong E^{r_0}_{p,q}$ for each $r \geq r_0$. ☐

We write $E^{\infty}_{p,q}$ for this stable value of $E^r_{p,q}$.

Definition 4.9. Let E be bounded. We say E converges to $H_\bullet$ if we are given a finite filtration of subobjects

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1} H_n \subseteq F_p H_n \subseteq \cdots \subseteq F_t H_n = H_n$$

such that $E^{\infty}_{p,q} = F_p H_{p+q}/F_{p-1} H_{p+q}$.

We denote this by $E^a_{p,q} \to H_{p+q}$.

Definition 4.10. A spectral sequence E collapses at E^r if there is only one non-zero row or column in E^r. If E collapses and converges to $H_\bullet$, we have $H_n = E^r_{p,q}$ where $p + q = n$.

Definition 4.11. (E^∞ terms) Let E be a spectral sequence. There is a sequence:

$$0 = B^a_{p,q} \subseteq \cdots \subseteq B^r_{p,q} \subseteq B^{r+1}_{p,q} \subseteq \cdots \subseteq Z^{r+1} \subseteq Z^r_{p,q} \subseteq \cdots \subseteq Z^a_{p,q} = E^a_{p,q}$$

such that $B^r_{p,q}/B^{r-1}_{p,q} \cong im(d^r)$, $Z^r_{p,q}/Z^{r+1}_{p,q} \cong ker(d^r)$, and $E^r_{p,q} \cong Z^r_{p,q}/B^r_{p,q}$. Introduce the intermediate objects:

$$B^\infty_{p,q} = \bigcup_{r=a}^{\infty} B^r_{p,q} \quad \text{and} \quad Z^\infty_{p,q} = \bigcap_{r=a}^{\infty} Z^r_{p,q}$$

and define $E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}$. This definition is compatible with the bounded case.

Remark 4.12. In an unbounded spectral sequence we tacitly assume that $B^\infty_{p,q}$, $Z^\infty_{p,q}$ and $E^\infty_{p,q}$ exist. It is true for the category of modules.

Definition 4.13. We say a spectral sequence E is bounded below if for every n there exists an integer $s(n)$, such that $p < s(n)$ implies $E^a_{p,n-p} = 0$. Dually change "$<$" into "$>$" for the cohomology case.

Definition 4.14. We say a spectral sequence E is regular if for each p, q there exists an integer r_0, such that $d^r_{p,q} = 0$ for every $r \geq r_0$.

Lemma 4.15. *Bounded below spectral sequences are regular.*

Proof. Let E be a bounded below spectral sequence. For each fixed p, q, choose r_0 such that $r_0 > p - s(p + q - 1)$. Thus for each $r \geq r_0$, $E^a_{p-r,q+r-1} = 0$. Then for every $r \geq r_0$ we have $E^r_{p-r,q+r-1} = 0$, which implies that the map $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1} = 0$ is just 0. ∎